Randomness Zoo

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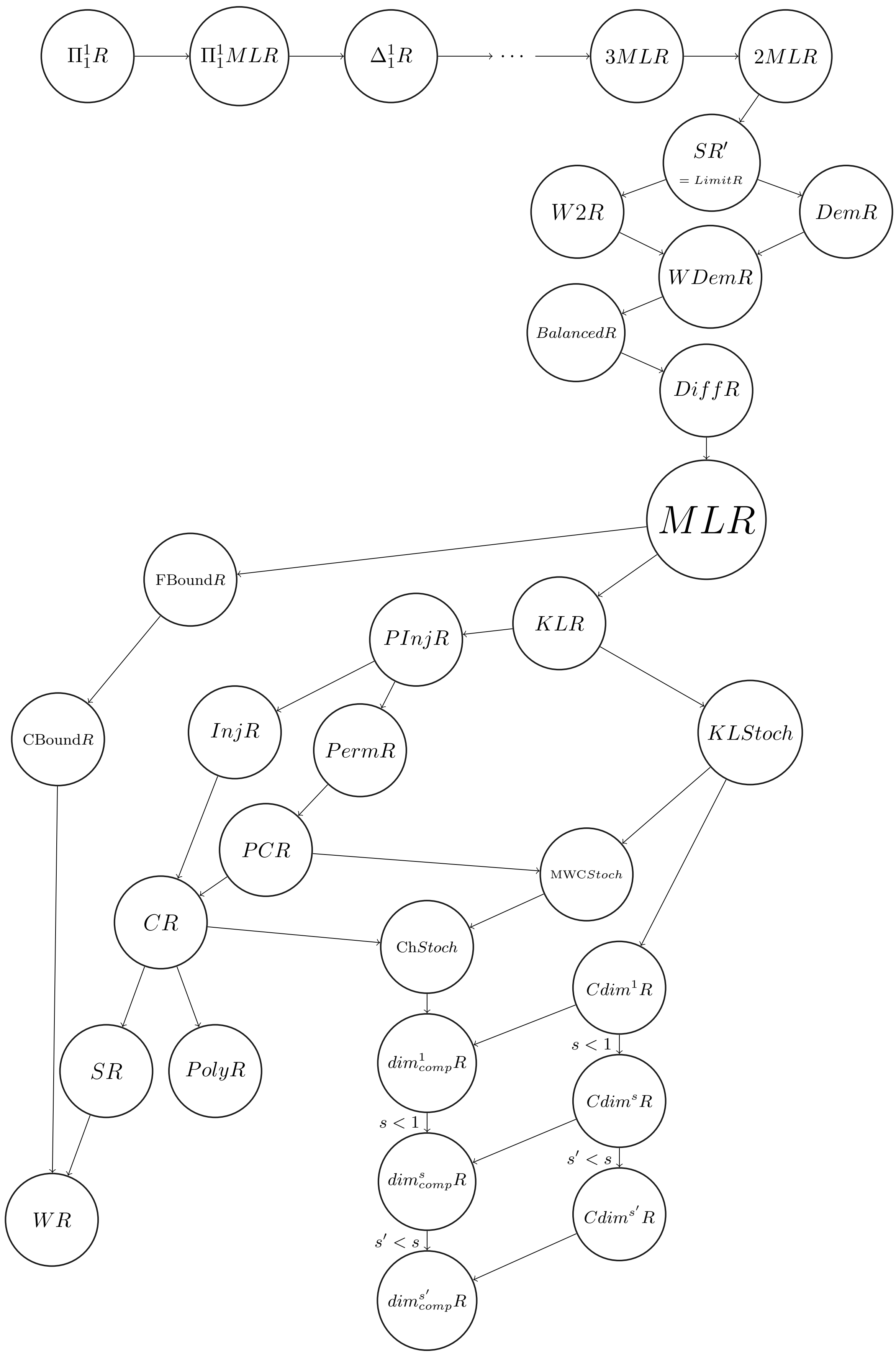

Definition 1 (**MLR**: Martin-Löf Randomness (Definition 3.2.1 in [ML66]))**.**

1. A Martin-Löf test, a ML-test for short, is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m) \leq 2^{-m}$.

2. A set R fails the test if $R \in \cap_{m\in\mathbb{N}} G_m$ otherwise R passes the test.

3. R is Martin-Löf Random if R passes each ML-test.

1 Weaker than MLR

Definition 2 (Scan rule function)**.** For a partial function $f : 2^{<\omega} \to \{\text{scan}, \text{select}\} \times \mathbb{N}$ we denote $n : 2^{<\omega} \to \mathbb{N}$ and $\delta : 2^{<\omega} \to \{\text{scan}, \text{select}\}$.

And we say that f is a scan rule if for all $\sigma, \rho \in 2^{<\omega}$ such that $\sigma \prec \rho$ we have $n(\sigma) \neq n(\rho)$.

The sequence of string observed by f on $Z \in cs$ is defined by(if f is well defined on this points):

$$V_f^A(0) = A(n(\varepsilon))$$

$$V_f^A(k+1) = V_f^A(k).A(n(V_f^A(k)))$$

and bits selected by f are:

$$T_f^A(0) = \varepsilon$$

$$T_f^A(n+1) = \begin{cases} T_f^A(n) & \text{if } \delta(V_f^A(n)) = \text{scan} \\ T_f^A(n).V_f^A(n+1) & \text{if } \delta(V_f^A(n)) = \text{select} \end{cases}$$

and we say that f is well defined on Z if $V_f^A(k)$ is well define for all k and $(T_f^A(n))$ converge to an infinite string and we denote this infinite string by T_f^A.

Definition 3 (Martingale)**.** A martingale is a function $\mathcal{M} : 2^{<\omega} \to \mathbb{R}^+ \cup 0$ such that for all $\sigma \in 2^{<\omega}$

$$\mathcal{M}(\sigma) = \frac{\mathcal{M}(\sigma 0) + \mathcal{M}(\sigma 1)}{2}$$

We say that $\mathcal{M}$ succeed on $Z \in 2^\omega$ if

$$\limsup_{n\to\infty} \mathcal{M}(Z \upharpoonright n) = \infty$$

Definition 4 (**KLR**: KolmogorovLoveland randomness (Definition in [Kol63] and [Lov66]))**.** $R \in 2^\omega$ is KL-random if for any partial computable scan rule function f and any partial computable martingale $\mathcal{M}$ such that f is well defined on R the martingale martingale $\mathcal{M}$ does not succeed on T_f^A.

Definition 5 (**KLStoch**: KolmogorovLoveland stochasticity (Definition in [Lov66])). Let $\#0 : 2^{<\omega} \to \mathbb{N}$ the function giving the number of "0" in a string.

A sequence R is KolmogorovLoveland stochastic if for all partial computable scan rule f such that f is well defined on Z we have:

$$\lim_{n\to\infty} \frac{\#0\left(T_f^A \upharpoonright n\right)}{n} = \frac{1}{2}$$

Definition 6 (**MWCStoch**: Mises-Wald-Church stochasticity (Definition in [vM19] and [Chu40])). A sequence R is Mises-Wald-Church stochastic if for all partial computable monotonic scan rule function f is well defined on Z we have:

$$\lim_{n\to\infty} \frac{\#0\left(T_f^A \upharpoonright n\right)}{n} = \frac{1}{2}$$

Definition 7 (**ChStoch**: Church stochasticity (Definition in [vM19] and [Chu40])). A sequence R is Church stochastic if for all total computable monotonic scan rule function f we have:

$$\lim_{n\to\infty} \frac{\#0\left(R_{f(0)}R_{f(1)}\dots R_{f(n)}\right)}{n} = \frac{1}{2}$$

Definition 8 (**PInjR**: partial injective randomness (Definition in [MN06])). A sequence R is partial injective random if for any total computable injective function $g : \mathbb{N} \to \mathbb{N}$ and any partial computable martingale $\mathcal{M}$ this martingale is defined and does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 9 (**InjR**: injective randomness (Definition in [**?**])). A sequence R is injective random if for any total computable injective function $g : \mathbb{N} \to \mathbb{N}$ and any total computable martingale $\mathcal{M}$ this martingale does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 10 (**PermR**: partial permutation randomness (Definition in [BHKM09])). A sequence R is partial permutation random if for any total computable bijective function $g : \mathbb{N} \to \mathbb{N}$ and any partial computable martingale $\mathcal{M}$ this martingale is defined and does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 11 (**PCR**: partial computable randomness (Definition in [AS97])). A sequence R is partial computable random if for all partial computable martingale $\mathcal{M}$ if $\mathcal{M}(R \upharpoonright n)$ is define for all n and $\mathcal{M}$ does not succeed on R.

Definition 12 (**CR**: computable randomness (Definition in [Sch71]))**.** A sequence R is computable random if for all total computable martingale $\mathcal{M}$ this martingale succeed on R.

Definition 13 (**SR**: Schnorr randomness(Definition in [Sch71]))**.** A Schnorr test is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m) = 2^{-m}$.

R is Schnorr random if for any Schnorr test $(G_m)_{m\in\mathbb{N}}$ $R \notin \cap_{m\in\mathbb{N}} G_m$

Definition 14 (**FBoundR**: finitely bounded randomness (Definition in [BDN12]))**.** R is finitely bounded random if R passes any Martin-Löf test (U_n) such that for every n, $\#U_n < \infty$ (with $\#U_n$ the number of sting enumerated in U_n).

Definition 15 (**CBoundR**: computably bounded randomness (Definition in [BDN12]))**.** A Martin-Löf test (U_n) is computably bounded if there is some total computable function f such that $\#U_n \leq f(n)$ for every n.

R is computably bounded random if R passes every computably bounded Martin-Löf test.

Definition 16 (**WR**: weakly randomness (Definition in [Kur81]))**.** R is weakly random if $R \in U$ for every Σ^0_1 set $U \subseteq 2^\omega$ of measure 1.

Definition 17 (**PolyR**: polynomial randomness (Definition in [Wan96]))**.** A sequence R is polynomially random if any martingale computable in polynomial time $\mathcal{M}$ do not succeed on R.

Definition 18 ($\mathbf{dim}^s_{\mathbf{comp}}\mathbf{R}$: computable s-randomness (Definition in [Lut03] and [May02]))**.** A computable s-test is a uniformly computable sequence $(G_m)_{m\in\mathbb{N}}$ of computable open sets such that for all n

$$\sum_{x\in G_m} 2^{-s|x|} \leq 2^{-n}.$$

R is computably s-random if for all $s' < s$ and computable s-tests (G_m) we have:

$$R \notin \bigcap_{m\in\mathbb{N}} G_m$$

Definition 19 ($\mathbf{Cdim}^s\mathbf{R}$: constructive s-randomness (Definition in [Lut03] and [May02]))**.** A constructive s-test is a uniformly computable sequence $(G_m)_{m\in\mathbb{N}}$ of computable enumerable open sets such that for all n

$$\sum_{x\in G_m} 2^{-s|x|} \leq 2^{-n}.$$

R is computably s-random if for all $s' < s$ and computable s-tests (G_m) we have:

$$R \notin \bigcap_{m \in \mathbb{N}} G_m$$

2 Stronger than MLR

Definition 20 (**DiffR**: difference randomness (Definition in [FN])). A difference test is given by a sequence $(V_m)_{m \in \mathbb{N}}$ of uniformly c.e. sets and a Π^0_1 set P such that $\mu(P \bigcap V_m) \leq 2^{-m}$ for every m.

A sequence R is difference random if for any difference test $(V_m)_{m \in \mathbb{N}}, P$ we have

$$R \notin P \bigcap \left(\cap_{m \in \mathbb{N}} V_m\right).$$

Definition 21 (**BalancedR**: balanced randomness (Definition in [FHM$^+$10])). A balanced test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some 2^n-c.e. function f and $\mu(V_m) \leq 2^{-m}$for every m.

A sequence R is balanced random if R passes any balanced test.

Definition 22 (**WDemR**: weak Demuth randomness (Definition in [Dem82])). Demuth test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some ω-c.e. function f and $\mu(V_m) \leq 2^{-m}$for every m.

A sequence R is weak Demuth random if for any Demuth test (V_m) we have $R \notin \cap_{m \in \mathbb{N}} V_m$.

Definition 23 (**DemR**: Demuth randomness (Definition in [Dem82])). A Demuth test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some ω-c.e. function f and $\mu(V_m) \leq 2^{-m}$for every i.

A sequence R is Demuth random if for any Demuth test (V_m) we have $R \notin V_m$ for almost all m.

Definition 24 (**W2R**: weak 2-randomness (Definition in [Kau91])). A sequence R is weak 2-random if $R \notin U$ for every Π^0_2 set $U \subset 2^\omega$ of measure 0.

Definition 25 (**LimitR**: limit randomness (Definition in [KN11])). A limit test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some Δ^0_2-computable function f and $\mu(V_m) \leq 2^{-m}$for every m.

A sequence R is limit random if for any limit test (V_m) we have $R \notin V_m$ for almost all m.

Definition 26 (Δ^1_1**R**: Δ^1_1 randomness (Definition in [ML70])). R is Δ^1_1-random if R avoids each null Δ^1_1-class.

Definition 27 (Π^1_1**MLR**: Π^1_1-Martin-Löf Randomness (Definition in [HN07]))**.** A Π^1_1-Martin-Löf test is a sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m) \leq 2^{-m}$ and the relation $\{\langle m,\sigma\rangle | [\sigma] \subseteq G_m\}$ is Π^1_1

R is Π^1_1-Martin-Löf Random if R passes each Π^1_1-ML-test.

Definition 28 (Π^1_1**R**: Π^1_1-Randomness (Definition in [HN07]))**.** R is Π^1_1-random if R avoids each null Π^1_1-class.

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